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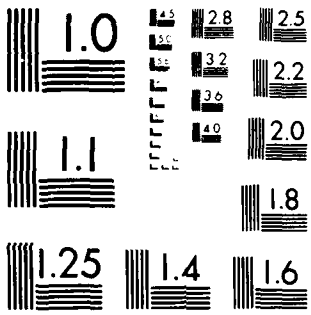

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LEAST SQUARES FOR FUZZY VECTOR DATA REGRESSION

AIVARS K. R. CELMINS

JUNE 1987

US ARMY BALLISTIC RESEARCH LABORATORY
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We consider model fitting problems in which the mathematical model of an observed event is a system of equations involving model parameters and observables, and in which the data are fuzzy vectors. Such problems naturally arise in applications when data are scarce and information is vague about distributions of variances that are contained in the observations. Then it may only be possible to obtain estimates of membership functions of the data. A model can be fitted to such data by maximizing the membership values of the adjusted observations. We achieve this by minimizing the sum of squares of the deviations of the membership values from one.

The optimization problem simplifies significantly if the membership functions of the fuzzy vectors belong to a class of conical functions, which are defined in terms of an elliptic norm. Properties of these functions are discussed and a membership propagation formula derived which component-wise obeys the extension principle.

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19. ABSTRACT (continued)

The least squares model fitting to the described fuzzy vector data deviates in important aspects from ordinary least squares. One obtains as a solution a fuzzy model parameter vector and a number of indicators which describe the quality of the fit, viz., the grade of compatibility between data and model, their discord and dilators of model and data membership functions. These new concepts are illustrated and discussed in the context of an example from a terminal ballistics problem.

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1. INTRODUCTION

We consider model fitting problems in which the data are fuzzy vectors, and the mathematical model of the observed event is a system of equations involving observables and model parameters. Such problems naturally arise when data are scarce and information is not available about distributions of data inaccuracies. Then it may only be possible to obtain estimates of data scatter from past experience in the form of estimated data membership functions.

The fitting of the model consists of finding an optimal set of parameters and a corresponding set of corrections to the observations such that the model equations are satisfied. One obtains different solutions, depending on the definition of optimality. In this article, the optimally is defined in terms of a least squares objective function. Particularly, we shall maximizing the sum of squares of the deviations of the membership values of one. One obtains relatively simple algorithms for this problem if the membership functions of the data vectors belong to a particular class of functions. This class of conical functions is defined and discussed in Section 3, after a short exposition of general fuzzy model fitting problems in Section 2. The model fitting to fuzzy vectors with conical membership functions is described in Section 4, and Section 5 presents an application example.

An obvious extension of the problem to fuzzy models is mentioned in Section 2, and will be treated in a forthcoming paper.

2. FUZZIFICATION OF MODEL FITTING PROBLEMS

We define in this section a general least squares model fitting problem and outline several possibilities how the problem can be fuzzified.

Let $X \epsilon R_n$ be the vector of all observations of an event, and let the mathematical model of the event be the equation

$$F(X, t) = 0 \qquad (2.1)$$

where F is a s-dimensional function, and t is a p-dimensional model parameter. We assume that F is twice differentiable with respect to all its arguments, and

$$n \geq s > p \geq 0 \tag{2.2}$$

The goal of a model fitting operation is to find such a parameter vector t that the model equation (2.1) is satisfied. Generally, this goal cannot be achieved, because the number s of equations in (2.1) usually is larger than the number p of free parameters. Then a solution of the problem only can be obtained by weakening the requirement that Eq. (2.1) be exactly satisfied. The replacement of Eq. (2.1) can be done in different ways and one obtains corresponding different solutions.

In usual least squares model fitting one assumes that the observations X may contain errors and, therefore, the model equations should not be satisfied at X, but in the vicinity of X. That is, one corrects the observations X by a term c_X and replaces the model equations (2.1) by the constraint equations

$$F(X + c_X, t) = 0 \tag{2.3}$$

One would like to have only small corrections and this is achieved by seeking such a solution c_X and t of Eq. (2.3) which minimizes the square of a norm of c_X. The problem may be formulated as follows:

$$\left.\begin{array}{ll} \text{minimize} & W = \| c_X \|^2 \\ \text{subject to} & F(X + c_X, t) = 0 \end{array}\right\} \tag{2.4}$$

Hence, a least squares model fitting problem is defined by the following set

$$\left\{ X ; \| c_X \| ; F ; W = \| c_X \|^2 \right\} , \tag{2.5}$$

containing a data vector, a definition of a norm, a definition of a model function, and a definition of an objective function as the square of the norm of corrections.

The problem (2.5) can be fuzzified as follows. First, the data X may be fuzzy. In this case one has in addition to the observed X also a definition of its membership function μ_X. Then we replace the definition (2.5) of the least squares model fitting problem by

$$\left\{ X \; ; \; \mu_X(c_X) \; ; \; F \; ; \; W = (1-\mu_X(c_X))^2 \right\} , \tag{2.6}$$

or

$$\left.\begin{array}{ll} \text{minimize} & W = (1-\mu_X(c_X))^2 \\ \text{subject to} & F(X+c_X, t) = 0 . \end{array}\right\} \tag{2.7}$$

This type of model fitting problems with fuzzy data is treated in Section 4.

Another possible fuzzification of the problem (2.5) is obtained by introducing a fuzzy model function. This can be done, for instance, by defining a set of functions with an associated membership function μ_F [1]. The data X then are assumed to be crisp, that is, one assumes $c_X=0$. The corresponding least squares model fitting problem is defined by the set

$$\left\{ X \; ; \; \| c_X \| = 0 \; ; \; F \; ; \; \mu_F \; ; \; W = (1-\mu_F)^2 \right\} , \tag{2.8}$$

or

$$\left.\begin{array}{ll} \text{minimize} & W = (1-\mu_F)^2 \\ \text{subject to} & F(X, t) = 0 . \end{array}\right\} \tag{2.9}$$

Problems of this type are akin to a problem formulated by Tanaka et al. [2] We shall treat the problem (2.8) in a forthcoming paper.

Another kind of fuzzification is a combination of the former two, that is, one assumes that the data are fuzzy and use a fuzzy function to fit the data. It is probably the most useful case for applications.

Finally, one may fuzzify the objective function W, for instance, by allowing to minimize other powers than the square of the norm or membership function. Because we restrict the present investigation to least squares minimization, we shall not further discuss this type of fuzzification.

3. FUZZY VECTOR SPACES AND SPREAD PROPAGATION FORMULAS

In many model fitting problems the data consist of a set of vectors rather than a set of scalars. Such data can be treated in the context of fuzzy sets by defining for each vector a membership function that depends on all

components of the vector. By chosing a proper class of membership functions one can simplify the numerical handling of the corresponding model fitting problem. We shall introduce and discuss in this section such a class of functions, namely a particular extension of the triangular membership function of a fuzzy scalar to n dimensions.

Let $\tilde{a}$ be a fuzzy scalar. We associate with it the triangular membership function $\mu_a(x)$, defined by

$$\mu_a(x) = \max\left\{0, 1 - |x-a| / s_a\right\} \tag{3.1}$$

We call a the *apex* and s_a the *spread* of $\tilde{a}$ and notice that $\mu_a(x)$ is completely determined by the two quantities a and s_a. The spread may be considered as a measure for the accuracy of $\tilde{a}$. Then a reasonable dimensionless measure for the difference between a crisp point x and the fuzzy observation $\tilde{a}$ is the norm

$$\|x - \tilde{a}\|_a = |x-a| / s_a \, . \tag{3.2}$$

The relation between this norm and the membership function of $\tilde{a}$ is

$$\mu_a(X) = 1 - \min\left\{1, \|x - \tilde{a}\|_a\right\} . \tag{3.3}$$

Now we extend this simple function to n dimensions and demand that the extension, that is, the membership function $\mu_A(X)$ of a fuzzy vector $\tilde{A}$, has the following properties:

(A) Each component of $\tilde{A}$ has a triangular membership function, and

(B) the function $(1-\mu_A(X))^2$ is differentiable in the domain where $\mu_A(X)>0$.

The latter property is important for the simplification of the numerical treatment of model fitting problems.

A function with the properties (A) and (B) is the conical membership function

$$\mu_A(X) = \max\left\{0, 1 - \left[(X-A)^T P_A^{-1} (X-A)\right]^{1/2}\right\}, \tag{3.4}$$

where P_A is diagonal matrix with diagonal elements which are the spread squares $s_{a_i}^2$ of the components $\tilde{a}_i$ of the vector $\tilde{A}$. Figure 1 shows $\mu_A(X)$ in the case $n=2$.

In analogy to Eq. (3.2) we also define a norm

$$\|X-\tilde{A}\|_A = \left[(X-A)^T P_A^{-1} (X-A)\right]^{1/2} \tag{3.5}$$

and obtain an expression of the membership function in terms of the norm:

$$\mu_A(X) = 1 - \min\left\{1, \|X-\tilde{A}\|_A\right\} . \tag{3.6}$$

The boundary of the support of the membership function is given by the equation

$$(X-A)^T P_A^{-1} (X-A) = 1 . \tag{3.7}$$

This boundary is a hyperellipsoid in the n-dimensional X-space. The principal axes of the ellipsoid are parallel to the coordinate axes, and their lengths are equal to $2s_{ai}$. The matrix P_A which defines the membership function, we call the ***panderance matrix*** of the fuzzy vector $\tilde{A}$.

It is practical to slightly generalize the given definition of a conical membership function by permitting the support ellipsoid to have principal axes that are not parallel to the coordinate axes. Such a situation arises, e.g., if one choses a different coordinate system. We determine the proper generalization of the definition by considering a linear coordinate transformation

$$Z = DX + Z_0 . \tag{3.8}$$

The apex A of the fuzzy vector $\tilde{A}$ with the panderance matrix P_A is mapped by this transformation into a vector

$$B = DA + Z_0 , \tag{3.9}$$

and the panderance matrix P_B of $\tilde{B}$ is obtained by the following spread propagation formula:

$$P_B = DP_A D^T . \tag{3.10}$$

We first verify this formula for the case where (3.8) is a regular coordinate transformation, that is, D is a non-singular $(n \times n)$-matrix. Then P_B, computed by Eq. (3.10), is positive definite and the membership function

$$\mu_B(Z) = 1 - \min\left\{1, \left[(Z-B)^T P_B^{-1} (Z-B)\right]^{1/2}\right\} \tag{3.11}$$

has the property

$$\mu_B(Z(X)) = \mu_A(X) , \tag{3.12}$$

because (3.13)

$$(Z-B)^T P_B^{-1}(Z-B) = (X-A)^T D^T (D^T)^{-1} P_A^{-1} D^{-1} D(X-A) = (X-A)^T P_A^{-1}(X-A).$$

In the Z-coordinate system, each component again has a triangular membership function, the projection of the cone defined by the panderance matrix P_B. The spread s_{bi} of each component $\tilde{b}_i$ of $\tilde{B}$ equals the square root of the i-th diagonal element of P_B. The support ellipsoid's principal axes are in general not parallel to the axes of the Z-coordinate system.

One sees from the formula (3.10) that the panderance matrix of a conical membership function merely has to be positive definite. However, as we shall see later, degenerated support ellipsoids do occur. Therefore we shall assume that the panderance matrix only is positive semidefinite.

The elements of a panderance matrix P_A have, in general, different physical dimensions, corresponding to the dimensions of the components of $\tilde{A}$. They also may have different orders of magnitude, depending on the units which are used to express A. Therefore, a more lucid representation of P_A is in terms of the spreads s_{ai} of the components $\tilde{a}_i$ of $\tilde{A}$ and a *concordance matrix* C_A, which is related to P_A by the formula

$$P_A = S_A C_A S_A ,$$

or

$$C_A = S_A^{-1} P_A S_A^{-1} , \tag{3.14}$$

where S_A is a diagonal matrix with spreads s_{ai} in the diagonal. Elementwise the relation (3.14) between panderances, concordances and spreads is

$$P_{Aij} = C_{Aij} s_{ai} s_{aj} , \quad i, j = 1, \ldots, n. \tag{3.15}$$

The concordance matrix has dimensionless elements, less or equal one in absolute value, with ones in the diagonal.

So far we only have verified the spread propagation formula (3.10) for regular linear coordinate transformations. It is, however, also valid for any other linear function (3.8) in the sense that the membership function which is produced by Eq. (3.10) for each component of Z satisfies the extension principle. Now we shall prove this property of the formula.

Let one component of the linear function (3.8) be

$$z = d^T X + z_0 . \tag{3.16}$$

The fuzzy vector $\tilde{A}$ is mapped into a fuzzy number $\tilde{b}$ with the apex

$$b = d^T A + z_0 \ . \tag{3.17}$$

The panderance (the spread square) of $\tilde{b}$ is according to (3.10)

$$s_b^2 = d^T P_A d \tag{3.18}$$

and the membership function of $\tilde{b}$ is

$$\mu_b(z) = 1 - \min\left(1, \left|z-b\right|/s_b\right) \ . \tag{3.19}$$

According to the extension principle, the membership function of should be

$$\mu_b(z) = \max_{X:\ d^T X + z_0 = z} \mu_A(X) \ . \tag{3.20}$$

We shall show that $\mu_b(z)$, computed from (3.20), equals the μ_b computed with the formula (3.19). To that end, we reformulate the right hand side of Eq. (3.20) as the following constrained minimization problem:

$$\left.\begin{array}{ll} \text{minimize} & \left\|X-\tilde{A}\right\|^2_A = (X-A)^T P_A^{-1}(X-A) \\ \text{subject to} & d^T(X-A) = z-b \ . \end{array}\right\} \tag{3.21}$$

We use a Lagrange multiplier k and obtain the following normal equations of the problem (3.21):

$$\left.\begin{array}{l} P_A^{-1}(X-A) - kd = 0 \ , \\ d^T(X-A) = z-b \ . \end{array}\right\} \tag{3.22}$$

We solve the first equation (3.22) for k, obtaining

$$k = d^T(X-A)/(d^T P_A d)$$

or

$$k = (z-b)/(d^T P_A d) \ . \tag{3.23}$$

Substituting this into the first equation (3.22) and multiplying from the left by $(X-A)^T$ one obtains

$$\left\|X-\tilde{A}\right\|^2_{A\,\min} = (z-b)^2/(d^T P_A d) \tag{3.24}$$

or

$$\left\|X-\tilde{A}\right\|_{A\,\min} = \left|z-b\right|/s_b \ . \tag{3.25}$$

Therefore,

$$\max \mu_A(X) = 1 - \min\left\{1, \left\|X-\tilde{A}\right\|_{A\,\min}\right\} = 1-\min\left\{1, \left|z-b\right| / s_b\right\}, \quad (3.26)$$

that is, the solution of (3.20) is (3.19).

The proof can be modified to cover also singular cases with semidefinite panderance matrices P_A by restricting the minimization (3.21) to the subspace where P_A is definite. Hence, we have shown that the spread propagation formula (3.10) componentwise satisfies the extension principle for any linear function of X. If the function $Z(X)$ is not linear, then one may obtain an approximate panderance matrix P_B for $B=Z(A)$ by linearizing the function. Then the propagation formula is

$$P_B = \frac{\partial Z}{\partial X} \; P_A \left(\frac{\partial Z}{\partial X}\right)^T , \quad (3.27)$$

where $\partial Z / \partial X$ is the Jacobian matrix of the function $Z(X)$. The quality of the approximation depends on the size of the second order derivatives of $Z(X)$, relative to the panderances and to the first order derivatives.

So far we only have considered single fuzzy vectors and defined a norm for the distance between a fuzzy and a crisp vector. The membership function was expressed by Eq. (3.6) in terms of this distance. We shall also need a measure for the distance between two fuzzy vectors. Now we give the necessary definitions and take for simplicity a general approach.

Let H_n be a vector space with fuzzy n-component vectors as elements. We assume that each $\tilde{A} \epsilon H_n$ is assigned a metric which defines a distance $h(X,\tilde{A})$ in R_n, that is, between any $X \epsilon R_n$ and $\tilde{A} \epsilon H_n$. We also assume that the membership function $\mu_{A(X)}$ of $\tilde{A}$ is related to $h(X,\tilde{A})$ by the formula

$$\mu_A(X) = \max\left\{0, 1-h(X,\tilde{A})\right\} = 1 - \min\left\{1, h(X,\tilde{A})\right\} . \quad (3.28)$$

The distances $h(X,\tilde{A})$ are not useful for the measurement of distances in H_n because in general $h(\tilde{A},B) \neq h(\tilde{B},A)$. We therefore, define a *discord* between two elements of H_n by the formula

$$D(\tilde{A},\tilde{B}) = \min_{X \epsilon R_n} \max\left\{h(X,\tilde{A}),\; h(X,\tilde{B})\right\} \quad (3.29)$$

D is symmetric but it is not a distance in H_n, that is, it has the properties

and

$$\left.\begin{array}{l} D(\tilde{A},\tilde{B}) > 0 \text{ if } A \neq B \\ D(\tilde{A},\tilde{B}) = 0 \text{ if } A = B \end{array}\right\} \qquad (3.30)$$

but it does not satisfy the triangle inequality.

As a complement to the discord $D(\tilde{A}, \tilde{B})$ we also define a *grade of collocation* of two fuzzy vectors $\tilde{A}$ and $\tilde{B}$ by

$$\gamma(\tilde{A},\tilde{B}) = \max_{X \in R_n} \min\left\{\mu_A(X), \mu_B(X)\right\} . \qquad (3.31)$$

Because we have assumed the relation (3.28) between μ_A and $h(X,\tilde{A})$, we also can express the grade of collocation in terms of the discord:

$$\gamma(\tilde{A},\tilde{B}) = \max\left\{0, 1-D(\tilde{A},\tilde{B})\right\} = 1-\min\left\{1, D(\tilde{A},\tilde{B})\right\} . \qquad (3.32)$$

In the special case of vectors with conical membership functions we define $h(X,\tilde{A}) = ||X-\tilde{A}||$. In this case, the distance $h(X,\tilde{A})$ is inversely proportional to the spreads of the components of A. If one multiplies the spreads of $\tilde{B}$ by a factor ϕ and lets ϕ approach zero, then one obtains for the discord the limit

$$\lim_{\phi \to 0} D(\tilde{B},\tilde{A}) = h(B,\tilde{A}) . \qquad (3.33)$$

The limit of a membership function with vanishing support may be considered as the membership function of a crisp vector. In this sense, the discord $D(\tilde{B},\tilde{A})$ approaches the distance $h(B,\tilde{A})$ between a crisp vector and $\tilde{A}$, as $\tilde{B}$ becomes a crisp B. Hence the discord is a convenient generalization of the distance in R_n with a proper limit property.

4. LEAST SQUARES MODEL FITTING TO FUZZY DATA

A general formulation of a least squares model fitting problem with fuzzy data was given in Section 2 by Eq. (2.7). Now we shall give a slightly different formulation which is more adequate for the present discussions.

Let the observations be fuzzy vectors $\tilde{X}_i$, $i=1,...,s$, each having the dimension n_i, and a membership function $\mu_{\tilde{X}_i}$, represented by a panderance matrix P_i. Let the model of the event by defined by the equations

$$F_i\,(X_i\,,\,t) = 0\,,\; i=1,\,.\,.\,.\,,s\,, \tag{4.1}$$

where t is a p-dimensional model parameter and F_i are r_i-dimensional functions. We assume that the dimensions n, r_i, and p satisfy the conditions

$$\sum_1^s n_i \geq \sum_1^s r_i > p \geq 0\;. \tag{4.2}$$

Compared to the formulation in Section 2 we now have assumed that the model equations have subsets which depend on distinct subsets of the observations. (The present formulation includes the formulation of Section 2 as the special case $s=1$) Such a partitioning is not always possible, but it offers substantial algorithmic advantages. [3] Also, many model fitting problems are naturally formulated by partitioned equations (4.1), e.g., in standard least squares problems where the F_i are scalar functions and all X_i are distinct.

A least squares model fitting problem now may be formulated as follows:

$$\left.\begin{aligned} &\text{minimize}\;\; W = \sum_{i=1}^{s}\left[1-\mu_{X_i}(X_i+c_i)\right]^2, \\ &\text{subject to}\;\; F_i\,(X_i+c_i\;,\,t) = 0\,,\; i=1,\,.\,.\,.\,,s\,, \end{aligned}\right\} \tag{4.3}$$

where

$$\mu_{X_i}(X_i+c_i) = 1-\min\left\{1\,,\,(c_i^T P_i^{-1} c_i)^{1/2}\right\}\,. \tag{4.4}$$

We recall the definition (3.5) of a norm

$$\|c_i\|_{X_i} = (c_i^T P_i^{-1} c_i)^{1/2} \tag{4.5}$$

and observe that the corrected observation X_i+c_i only is meaningful if the correction c_i satisfies the inequality $||c_i||_{X_i} < 1$. If $||c_i||_{X_i} \geq 1$ then the membership function $\mu_{X_i}(X_i+c_i)$ vanishes and in general the function $F_i(X_i+c_i,t)$ might not even be defined. Therefore, one should supplement the constraints of the minimization problem with the conditions $||c_i||_{X_i} < 1-\overset{*}{\gamma}$ with a $\overset{*}{\gamma}$ satisfying $0<\overset{*}{\gamma}<1$.

We also express the objective function in terms of the $||c_i||_{X_i}^2$ using the relation (4.4). The advantage of this reformulation is that the new objective function is differentiable with respect to the c_i for all finite c_i. As a result one can use Lagrange multiplier techniques for the numerical solution of the problem. The restriction to conical membership functions was made in order to have such a differentiable objective function. Using the new objective function and the above discussed additional constraints one obtains the following constrained minimization problem:

$$\left.\begin{array}{ll} \text{minimize} & W = \sum_1^s c_i^T P_i^{-1} c_i \,, \\ \text{subject to} & F_i(X_i+c_i,t) = 0, \; i=1,\ldots,s \end{array}\right\} \tag{4.6}$$

$$\text{and} \qquad \|c_i\|_{X_i} < 1-\overset{*}{\gamma}\,, \; i=1,\ldots,s\,. \tag{4.7}$$

One observes that the formal difference between an ordinary least squares problem and a model fitting with fuzzy data are the constraints (4.7). Conceptually, these constraints are compatibility conditions between data and model, and they identify outliers. Geometrically, they require that for all $i=1,\ldots,s$ the

support of the function $\mu_{\tilde{X}_i}(X) - \overset{*}{\gamma}$ intersects with the fitted surface $F_i(X,t)=0$. However, the results of the fitting is not a crisp function, but a fuzzy set of functions, because it has been computed from fuzzy data. Therefore, the fitted surface also is fuzzy. In the X-space it has a membership function $\mu_{F_i}(X)$, which equals one on the surface $F_i=0$, and has a finite support around that surface. The condition (4.7) means that the support of the functions $\mu_{\tilde{X}_i}(X) - \overset{*}{\gamma}$ and $\mu_{F_i}(X) - 1$ are required to intersect. It is more reasonable to only require that the supports of $\mu_{\tilde{X}_i}(X) - \overset{*}{\gamma}$ and $\mu_{F_i}(X) - \overset{*}{\gamma}$ intersect. Now we shall formulate a condition based on this requirement, and replace (4.7) by this condition.

First, we compute the membership function of the fitted function $\tilde{F}_i$. Let P_t be the panderance matrix of the fuzzy model parameter vector $\tilde{t}$. (One obtains an estimate of P_t concurrently with the numerical solution of the minimization problem (4.6), see Ref 3). The panderance P_{F_i} of F_i is approximately given by Eq. (3.27), that is, by

$$P_{Fi} = F_{ti} P_t F_{ti}^T , \tag{4.8}$$

where $F_{ti}=\partial F_i/\partial t$. We assume that the r_i components of F_i are independent funtions of t and that $r_i \leq p$, so that the rank of F_{ti} equals r_i. Then P_{F_i} is positive definite, and defines the following norm for F_i :

$$\left\| F_i(X) \right\|_{Fi} = \left(F_i^T P_{Fi}^{-1} F_i \right)^{1/2}, \tag{4.9}$$

and the corresponding membership function in the X-space

$$\mu_{Fi}(X) = 1 - \min\left\{ 1 , \left\| F_i(X) \right\|_{Fi} \right\} . \tag{4.10}$$

Next, we define the *grade of compatibility* between the observation $\tilde{X}_i$ and the fitted model by

$$\gamma_{XFi} = \max_{X \in R_{ni}} \min\left\{ \mu_i(X), \mu_{Fi}(X) \right\} . \tag{4.11}$$

The constraint (4.7) of the minimization problem (4.6) should be replaced by the requirement that the grade of compatibility is for each observation higher than $\overset{*}{\gamma}$, that is, by

$$\gamma_{XFi} > \overset{*}{\gamma} , \quad i = 1, \ldots , s . \tag{4.12}$$

In this form the constraints may not be easily implemented, but in most cases one can approximate them by conditions on the grade of collocation, Eq. (3.31), between the observations $\tilde{X}_i$ and the corrected observations $\tilde{X}_i+\tilde{c}_i$. Using Eq. (3.32) we shall express these conditions in terms of the discord between these two vectors. We use $||c_i||_{Xi}$ as a distance measure $h(\tilde{X}_i+c_i,\tilde{X}_i)$ from $\tilde{X}_i$ and a linearized form of $||F_i||_{Fi}$ as a distance measure $h(X,\tilde{X}_i+\tilde{c}_i)$ from $\tilde{X}_i+\tilde{c}_i$. Now we shall derive this linearized form. Let Y_0 be a point on the surface $F_i = 0$. Then to the first order

$$\Delta F_i = F_i(X_0+b,t) - F_i(X_0,t) = F_i(X_0+b,t) = F_{xi}\ b \tag{4.13}$$

where $F_{Xi}=\partial F_i(X_0,t)/\partial X$. Substituting this expression for F_i into Eq. (4.9) one obtains

$$\|\Delta F_i\| = (b^T F_{xi}^T P_{Fi}^{-1} F_{xi}\ b)^{1/2} \quad . \tag{4.14}$$

Let

$$P_b^{-1} = F_{xi}^T P_{Fi}^{-1} F_{xi} \quad . \tag{4.15}$$

P_b^{-1} is a $n_i \times n_i$-matrix and, because in general $n_i > r_i$ (unless the model is a point) the matrix is only semidefinite. Therefore, the surfaces $b^T P_b^{-1} b = const$ are degenerated ellipsoids in the X-space, namely, cylinders with rulings that are parallel to $F_i = 0$. The pseudonorm $||b||_{Fi}=(b^T P_b^{-1} b)^{1/2}$ only measures that component of b which is orthogonal to $F_i = 0$. (Orthogonal in the sense of the inner product associated with the pseudonorm). The use of the linearized pseudonorm $||b||_{Fi}$ instead of $||\Delta F_i||_{Fi}$ in the definition (4.10) of the membership function μ_{Fi} amounts to a linearization of μ_{Fi} in the X-space. Now we use $||b||_{Fi}$ as a measure for the distance in X-space from the corrected observation, that is,

$$h(X_i+c_i+b,\ \tilde{X}_i+\tilde{c}_i) = \|b\|_{Fi} \quad . \tag{4.16}$$

With these definitions, the discord between $\tilde{X}_i$ and $\tilde{X}_i+\tilde{c}_i$ is according to Eq. (3.29)

$$D_i = \min_{X \in R_{ni}} \ \max\left\{\|c\|_{Xi}, \|b\|_{Fi}\right\} \tag{4.17}$$

where $e = X - X_i$ and $b = X - (X_i + e_i)$

If the linearization (4.14) is sufficiently accurate and the dimension of F_i equals one then the value of the discord D_i can be explicitly calculated from the two norms of the residual $||e_i||_{X_i}$ and $||e_i||_{F_i}$, because the minimum in Eq. (4.17) is attained at a point on the straight line segment between X_i and X_i and e_i. We demonstate this property of the minimum by observing that e_{X_i} is a convex function and showing that the level planes $b_{F_i} = const$ are tangential to the level surfaces of $e_{X_i} = const$ for all points of the segment. A plane $b_{F_i} = const$ is spanned by vectors q which satisfy the equation

$$F_{xi}\, q = 0 \tag{4.18}$$

(See the definition Eq. (4.14).) A plane through the point $X_i + e$ and tangential to $||e||_{X_i} = const$ is spanned by vectors $\bar{q}$ which satisfy the equation

$$e^T P_i^{-1} \bar{q} = 0 \tag{4.19}$$

(See the definition Eq. (4.5)). Let $X_i + e$ be on the straight line through X_i and $X_i + e_i$ i.e., $e = \alpha e_i$. The residual is a solution of the least squares problem (4.6). Therefore, it satisfies the following equation from Reference 4.

$$e_i = P_i F_{xi}^T (F_{xi} P_i F_{xi}^T)^{-1} F_{xi}\, e_i \quad . \tag{4.20}$$

Substituting α times the right hand side of Eq. (4.20) for e in Eq. (4.19), one obtains

$$e_i^T F_{xi}^T (F_{xi} P_i F_{xi}^T)^{-1} F_{xi}\, \bar{q} = 0 \tag{4.21}$$

The set of all vectors $\bar{q}$ satisfying Eq. (4.21) are in the plane tangential to $||e||_{X_i} = const$. Because q satisfies Eq. (4.18), it also satisfies Eq. (4.21) and, is therefore, a tangent vector. Hence $||b||_{F_i} = const$ is tangential to $||e||_{X_i} = const$.

Now we compute the discord D_i, taking in Eq. (4.17) the maximum only over points of the straigt line between X_i and $X_i + e_i$. On that line, $e = \alpha e_i$ and $b = (\alpha - 1) e_i$. Therefore, Eq. (4.17) becomes

$$D_i = \min_{\alpha} \max\left\{ |\alpha| \, \|c_i\|_{X_i} \, , \, |1-\alpha| \, \|c_i\|_{F_i} \right\} \qquad (4.22)$$

which has the solution

$$D_i = \|c_i\|_{X_i} \, \|c_i\|_{F_i} / \left(\|c_i\|_{X_i} + \|c_i\|_{F_i} \right) . \qquad (4.23)$$

The corresponding grade of collocation between X_i and X_i+c_i is

$$\gamma_i = \max\left\{ 0, 1-D_i \right\} . \qquad (4.24)$$

We use this γ_i as an approximation to γ_{XF_i} and replace the compatibility condition (4.12) by

$$\gamma_i > \overset{*}{\gamma} , \quad i = 1, \ldots, s . \qquad (4.25)$$

With the same approximation we define the overall discord between data and model as

$$D = \max_{i=1,\ldots,s} D_i , \qquad (4.26)$$

and the *overall grade of compatibility* as

$$\gamma = \min_i \gamma_i = \max\left\{ 0, 1-D \right\} \qquad (4.27)$$

The reformulated least squares model fitting problem (4.6) and (4.7) is:

$$\left.\begin{array}{ll} \text{minimize} & W = \sum_{i=1}^{s} c_i^T P_i^{-1} c_i \\ \text{subject to} & F_i(X_i + c_i , t) = 0 , \quad i = 1, 2, \ldots, s \end{array}\right\} \qquad (4.28)$$

$$\text{and} \qquad \|c_i\|_{X_i} \, \|c_i\|_{F_i} / \left(\|c_i\|_{X_i} + \|c_i\|_{F_i} \right) < 1 - \overset{*}{\gamma} . \qquad (4.29)$$

The numerical solution of the model fitting problem may be obtained by first solving Eq. (4.28), and checking the conditions (4.29) after solution. If the conditions are satisfied, then one has the solution. If for some X_i the compatibility grade is less than the required $\overset{*}{\gamma}$, then one may take several actions. First one should recheck the data and depending on the circumstances, one

may discard outliers, lower the required compatibility grade $\overset{*}{\gamma}$, modify the spread estimates of observations and/or model, change the model, or try to find a solution that does satisfy Eq. (4.29). The latter means, however, an accommodation of outliers at the expense of the overall fit. A solution will not always exist and, if it does exist, it usually will not be desirable. Discarding of observations and changing the model equations are problem dependent actions and will not be discussed here. Changing of $\overset{*}{\gamma}$ merely changes the right hand side of Eq. (4.29). The more general actions are the modifications of the spread estimates which we now shall discuss.

We first consider the effect of changing all spread estimates of the observations $\tilde{X}_i$ by a factor ϕ_X. Then all panderance matrices P_i are multiplied by ϕ_X^2 and $||c_i||_{Fi}$ as well $||c_i||_{Fi}$ are multiplied by $1/\phi_X$. (P_t in Eq. (4.8) receives the factor ϕ_X^2, which affects $||b||_{Fi}$ through Eq. (4.14)). Consequently, the discords D_i in Eq. (4.23) are multiplied by $1/\phi_X$, too. Hence in order to make all discords less than $1 - \overset{*}{\gamma}$, i.e., all compatibility grades larger than $\overset{*}{\gamma}$, one may multiply the estimated data spreads by

$$\phi_X = \max_i \left\{ D_i/(1-\overset{*}{\gamma}) \right\}. \tag{4.30}$$

We call ϕ_X the *data spread dilator*. If the conditions (4.29) are satisfied, then the data spread dilator is less or equal one. If the compatibility grade γ between data and model is less than $\overset{*}{\gamma}$, then the data spread dilator ϕ_X provides a measure for the seriousness of maximum deviation from $\overset{*}{\gamma}$. The individual ϕ_{Xi} i.e., the right hand side terms of Eq. (4.30), provide such a measure for each observation X_i.

Now we consider model spread dilators. One may use such a dilator if the compatibility grade γ between data and model is less than the desired $\overset{*}{\gamma}$, but the data spread estimates and the model equations are firmly established. Then one may achieve the desired grade of compatibility by increasing the computed fuzziness of the model, i.e., by dilating the spread of the model. In general, this can be done by modeling the spreads of the model function concurrently with the modeling of the event. We consider here the simplest case of such a modeling, namely the multiplication of the panderance matrix P_t in Eq. (4.8) by a factor. A value of the factor may be obtained from one of the following two requirements. First, one obtains a *minimal model spread dilator* ϕ_F by the requirement that the constraints (4.29) be satisfied. This dilator is a complement to the data spread dilator ϕ_X. We obtain its value by observing that the multiplication of the model parameter spreads by ϕ_F changes $||c_i||_{Fi}$ in Eq. (4.29) by the factor $1/\phi_F$, and leaves the $||c_i||_{Xi}$ unchanged. Then all

constraints (4.29) can be satisfied if ϕ_F has the value.

$$\phi_F = \max_i \max\left\{0, \|c_i\|_{Fi} / (1-\overset{*}{\gamma}) - \|c_i\|_{Fi} / \|c_i\|_{Xi}\right\} . \qquad (4.31)$$

ϕ_F is less or equal one if all $D_i < 1-\overset{*}{\gamma}$, i.e., if the original model is compatible with the data to the desired grade. Otherwise ϕ_F has a value bigger than one. The individual ϕ_{Fi}, i.e., the terms on the right hand side of Eq. (4.31) again provide a measure of the seriousness of the incompatibility between the observation $\tilde{X}_i$ and the fitted model. If the support of the function $\mu_{Xi} - \overset{*}{\gamma}$ intersects with the surface $F_i = 0$ then ϕ_{Fi} equals zero.

Another model spread dilator is more practical for application. We call it the *inclusive model spread dilator* Φ_F and determine its value from the requirement that the support of $\mu_{Xi} - \overset{*}{\gamma}$ should include all support ellipsoids of the functions $\mu_{Fi} - \overset{*}{\gamma}$. We obtain an approximate value of Φ_F by the following algorithm. First, we find for each observation $\tilde{X}_i$, that point on the straight line through X_i and $X_i + c_i$ which has the membership value $\overset{*}{\gamma}$ and which is on the side of X_i opposite to $X_i + c_i$. That point has the coordinates

$$Y_i = X_i + c_i (1-\overset{*}{\gamma}) / \|c_i\|_{Xi} . \qquad (4.32)$$

The membership value of F_i at that point is

$$\mu_{Fi}(Y_i) = 1 - \min\left\{1, \|F_i(Y_i)\|_{Fi}\right\} , \qquad (4.33)$$

where

$$\|F_i(Y_i)\|_{Fi} = \left(F_i(Y_i)^T P_{Fi}^{-1} F_i(Y_i)\right)^{1/2}$$

is calculated by Eq. (4.9). We want this value to be equal or larger than $\overset{*}{\gamma}$. Therefore,

the spread of the model parameter vector should be multiplied by

$$\Phi_{Fi} = \|F_i(Y_i)\|_{Fi} / (1 - \overset{*}{\gamma}) . \tag{4.34}$$

The overall inclusive model spread dilator is

$$\Phi_F = \max_i \Phi_{Fi} . \tag{4.35}$$

In summary, we have introduced the following indicators which measure the quality of the model fitting:

D - discord between data and model,
γ - grade of compatibility between data and model,
ϕ_X - data spread dilator,
ϕ_F - minimal model spread dilator,
Φ_F - inclusive model spread dilator.

The last three indicators depend on a desired minimum grade of compatibility, whereas the first two indicators are independent of such a grade.

5. EXAMPLE

As an example of a model with fuzzy experimental data we present the following treatment of perforation of a plate by projectiles. In perforation experiments, one is interested in the minimum striking velocity which is needed to achieve perforation. That velocity is called the ballistic limit velocity and for a given target it depends on the materials of the projectile and target, on the form of the projectile, and on the obliquity of the impact. The limit velocity typically is determined by firing projectiles with different velocities and

observing their residual velocities (the velocity after penetration). The limit velocity is computed from these data by one of several available methods which all amount to some kind of interpolations between zero and non-zero residual velocities. The result contains observational errors as well as intrinsic variations of the limit velocity between rounds due to unknown causes, e.g., due to variations of target and projectile material properties. (Such variations manifest themselves e.g., by overlapping regions of zero and non-zero residual velocities.) The number of experiments usually is too small to determine a distribution of the result, but expert estimates of possible variations are available from past experience. Table I contains a set of observed limit velocities. (The data are taken from Ref. 5). The targets were identical in all cases. The projectiles were cylindrical rods with different nose shapes, quantitated in the table by a "shape quotient" which is the ratio of nose length to rod length. Because it is not certain that this ratio adequately describes the significance of the projectile's geometry, a substantial spread of 0.05 was assigned to the value of the ratio. The spread of the limit velocity was estimated from past experience to be about 50 m/s. However, in two cases the actual observations were not sufficient to guarantee such an accuracy, and, therefore, corresponding larger spreads were assumed in these cases.

The projectiles were fired in two modes: normal to the target plate, and at a constant angle of obliquity. The purpose of the experiments was to determine whether and how the shape of the projectile affects the limit velocity. The problem was handled by fitting a constant and a linear function to the two series of observed limit velocities, and by comparing the results. The numerical fitting of the fuzzy data was done with the general least squares computer program COLSAC [6].

The data are in the present case two-component fuzzy vectors $(\tilde{Q}, \tilde{V})$. The component $\tilde{Q}$ is the dimensionless shape quotient and the component $\tilde{V}$ is the corresponding limit velocity in m/s. The models of the limit velocity were the linear functions

$$V = a + bQ \ . \tag{5.1}$$

Using the technique described in Section 4, one obtains as a result of each model fitting a fuzzy parameter vector with the components $\tilde{a}$ (m/s) and $\tilde{b}$ (m/s), and a corresponding (2×2) panderance matrix. In case of a constant model one sets $b=0$ in Eq. (5.1) ($\tilde{b}$ is a crisp zero in that case), and the model fitting algorithm provides only $\tilde{a}$ and its panderance. In the (a,b)-space, the special solution vector $(\tilde{a},0)$ with a crisp second component has a singular

panderance matrix. The support of the corresponding membership function is a segment of the a-axis. The numerical values of the four fuzzy parameter vectors are listed in Table II. The data and the fitted curves are shown in Figures 2 through 7.

Next, we discuss and illustrate the various measures of compatibility between data and models. Table II lists these indicators for each data set $(\tilde{Q}, \tilde{V})$ from the normal impact observations series. (See also Figures 2 and 4). To be specific we have assumed that the desired grade of compatibility is $\overset{*}{\gamma}=0.3$, and Table III shows that the normal impact data exceed this level of compatibility with the constant model as well as with the linear model. Consequently the data dilators and the minimal model dilators all are less than one.

Figure 2 shows the normal impact data and the fitted constant model. Each data point is plotted with the support ellipse of its membership function. A dotted ellipse indicates the level line $\mu_X = 0.3$. The fitted velocity model is the line $V = 819.2$ m/s. It is a fuzzy line, with a spread of 24.5 m/s, as indicated in Figure 2. The dotted horizontal lines are the level lines $\mu_X = 0.3$. The data dilators and the minimal model dilators all are less than one (see Table III) because the supports of μ_{X_i}- 0.3 and μ_F- 0.3 intersect. The inclusive model dilators in Table III for the desired grade of compatibility $\overset{*}{\gamma} = 0.3$ are larger than one because the supports of μ_{X_i}- 0.3 are not subsets of the support μ_F- 0.3. Similarly, the supports of μ_{X_i} are not subsets of the support of μ_F, and therefore, also the inclusive model dilators for $\overset{*}{\gamma} = 0$ ought to be larger than one. Their maximum value $\Phi_0 = 6.832$ is listed in Table II. If the spread of the model is dilated by the factors Φ_0 or $\Phi_{0.3}$, then it includes the supports of the data up to the corresponding level 0.0 or 0.3, respectively.

Figure 3 shows the same data as Figure 2 but the fitted model is a linear function. The discords between data and fitted model are smaller than for the constant model (see Table III). The inclusive model dilators have reasonable values, $\Phi_0 = 2.290$ for $\overset{*}{\gamma} = 0.0$ and $\Phi_{0.3} = 2.407$ for $\overset{*}{\gamma} = 0.3$. Figure 4 shows the effect of these dilators. It displays the same data and model as Figure 3, but the model spread has been dilated. The solid line indicates the boundary of the support of μ_F, if the dilator $\Phi_0 = 2.290$ is used. The dotted lines indicate the boundary of the support of μ_F and the level line $\mu_F = 0.3$ if the dilator $\Phi_{0.3}=2.407$ is used. In the former case, the support of μ_F includes all supports of the data. In the second case, the support of μ_F- 0.3 includes the supports of μ_{X_i}- 0.3. Figure 4 confirms that the approximately calculated dilators by Eq. (4.34) are sufficiently accurate in this example.

Next, we consider the oblique impact data listed in Table IV. The table indicates that the data are incompatible with a constant limit velocity model. and Figure 5 shows that the supports of the data indeed do not intersect with

the support of the fitted model, except for one data point. All dilators are larger than one, particularly the inclusive model dilators Φ_0=11.579 and $\Phi_{0.3}$=14.941. (Table II).

Figure 6 shows the result of fitting a linear function to the oblique impact data. The data set is compatible with the model, the grade of compatibility is 0.298. Consequently, all data dilators and minimal model dilators for the desired grade of compatibility $\overset{*}{\gamma}=0.3$ are less than one or close to one (Table IV). The inclusive model dilators are somewhat large, $\Phi_0 = 4.625$ and $\Phi_{0.3}$=5.464. Using these dilators to increase the spread of the fitted function one obtains the result shown in Figure 7. As in Figure 4, we have plotted as solid curves the level lines $\mu_F = 0$, dilated by Φ_0. For the spread of the fitted function, dilated by $\Phi_{0.3}$ we have plotted as dotted curves the level lines $\mu_F = 0.0$ and $\mu_F = 0.3$. The regions between these curves should include all corresponding supports of the data membership functions. One notices a slight intersection of the boundary curves of the second data point (Q=0.052) with the model spread boundaries. This intersection (instead of a tangent) is caused by the approximate calculation of the dilators by Eq. (4.34), only using function values along the line through the observation X and the corrected observation $X+c$.

One can conclude from the presented results that in case of oblique impact the limit velocity indeed depends on the nose shape, increasing with the slenderness of the projectile's nose. In the normal impact case there seems to be a slight decrease of the limit velocity with increasing nose shape quotient. The data are compatible with the constant model to a grade 0.640 and compatible with the linear model and a grade 0.859. The increase of the compatibility is not very large and one might ask to what degree both fitted models differ from each other. A measure for this difference is the grade of collocation between the two fuzzy model parameter vectors which define the models (Table II). Figure 8 shows the two vectors with the corresponding supports of their membership functions in the parameter space. As mentioned before, the support of the parameter $(\tilde{a},0)$, i.e., the constant model parameter, consists of a segment of the a-axis. The support of the linear model parameter membership function is an ellipse, shown in Figure 8 with a solid line. The dotted line curve in the figure is the level line $\mu = 0.3$. The grade of collocation between these two vectors turns out to be 0.080. If one increases the spread of the parameters by the factors Φ_0 or $\Phi_{0.3}$, then one also increases the grade of collocation, namely to 0.598 and 0.618, respectively.

For the oblique impact data the model vectors of constant and linear models have a zero grade of collocation. Figure 9 shows the situation of the vectors in the parameter space. The discord between the vectors is in this case

3.250. (If one uses the dilators Φ_0 or $\Phi_{0.3}$, then the grades of collocation are 0.296 and 0.404, respectively.) The large discord is an indication that there is a significant difference between both models. In such a situation one should use the better model to represent the data, in this case the linear model. This leads to the same conclusion as above: for oblique impact, the limit velocity does increase with the slenderness of the projectile.

We make a final comment to the representation of the data by a model, and consider as an example the oblique impact data. A straight forward determination of the model produces the result, shown in Figure 6, which is compatible with the data, but does not include all supports of the data. Using model dilators one obtains the results shown in Figure 7, which includes all data supports, but seems to be excessively pessimistic at some parts of the fitted curve. Hence, if one intends to use the fitted model for predictive purposes, one may also want to model the spread of the results so that it reflects the available data more closely. Such a modeling depends very much on the particular application. In the present example, using a spread model that is of second degree in Q, and symmetric about the fitted line one obtains the result shown in Figure 10. It is probably more reasonable for predictive purposes than the result shown in Figure 7.

6. SUMMARY AND CONCLUSIONS

This article exposes several possible fuzzifications of model fitting, and treats in detail the fitting of a crisp model to fuzzy vector data. The result is a fuzzy fitted model, whereby the fuzziness of the model depends of the fuzziness of the data.

The data type is specialized to fuzzy vectors with conical membership functions. The restriction to conical functions greatly facilitates the numerical treatment of the problem, and a simple spread propagation formula can be derived for such functions. A new concept is the discord between two fuzzy vectors. It is a dimensionless and symmetric measure of the separation of the vectors. A complementary new concept is the grade of collocation of two elements of a fuzzy vector space.

The restriction to conical membership functions permits one to use

available model fitting computer programs with little changes. The quality of the fit is characterized by a discord between data and fitted model and a grade of compatibility between data and model. Both concepts naturally arise from the problem formulation. If the grade of compatibility is not as high as desired, then one can either try a different model, or increase the fuzziness of the data and/or of the fitted model, such that it becomes compatible with the data. The factors which achieve this are defined as data and model spread dilators, respectively, and explicit approximate formulas are derived for their calculation. The values of the dilators depend on the desired grade of compatibility, as well as on the data and their membership functions.

The application of model fitting to fuzzy data is likely to be in problems with scarce data and little information about data distribution. The description of the observed event by a fuzzy function, as derived in this paper, explicitly exposes the lack of exact information. A proper quantitation of this deficiency is achieved by the use of model dilators. The restriction of the algorithms to conical membership functions is likely not a serious impediment: because of the simplicity of the algorithms one easily can carry out the model fitting with different estimates of data spreads and, comparing the results, determine the significance of such estimates.

In summary, the described model fitting provides realistic representation of events about which only fuzzy information is available.

Table I. Ballistic Limit Data

Projectile Type	Shape Quotient Q	Spread s_Q	Normal Impact Limit Velocity V (m/s)	Normal Impact Spread s_V (m/s)	Oblique Impact Limit Velocity V (m/s)	Oblique Impact Spread s_V (m/s)
Flat	0.000	0.050	841	50	1060	50
Hemispherical	0.052	0.050	846	50	978	90
Short Cone	0.165	0.050	803	50	1111	50
Long Cone	0.232	0.050	795	50	1255	50
Full Cone	0.431	0.050	772	120	1300	50

Table II. Parameters of Fitted Models

Model $V = a + bQ$

Normal Impact

Model Type	a	s_a	b	s_b	c_{ab}	Φ_0	$\Phi_{0.3}$
Constant	819.2	24.5	0.0	0.0	0.0	6.832	7.658
Linear	845.3	37.9	-207.8	226.0	-0.752526	2.290	2.407

Oblique Impact

Model Type	a	s_a	b	s_b	c_{ab}	Φ_0	$\Phi_{0.3}$
Constant	1166.9	24.1	0.0	0.0	0.0	11.579	14.941
Linear	1030.2	51.2	688.7	211.4	-0.824466	4.625	5.464

The dilators Φ are not included in the quoted values of the spreads s_a and s_b. The c_{ab} are the concordances between the components a and b of the parameter vectors. The units of a, s_a, b and s_b are m/s.

Table III. Model Fitting Results for Normal Impact Data Constant Model

Constant Model

Projectile	$\|c\|_X$	$\|c\|_F$	Discord	Comp. grade	Data Dilator	Model Dilators Minimal	Model Dilators Inclusive
Flat	0.436	0.891	0.293	0.707	0.418	0.0	3.315
Hemispherical	0.536	1.095	0.360	0.640	0.514	0.0	3.607
Short Cone	0.324	0.662	0.218	0.789	0.311	0.0	2.989
Long Cone	0.484	0.989	0.325	0.675	0.464	0.0	3.456
Full Cone	0.393	1.928	0.327	0.673	0.467	0.0	7.658

Linear Model

Projectile	$\|c\|_X$	$\|c\|_F$	Discord	Comp. grade	Data Dilator	Model Dilators Minimal	Model Dilators Inclusive
Flat	0.084	0.114	0.048	0.952	0.069	0.0	1.461
Hemispherical	0.225	0.378	0.141	0.859	0.202	0.0	2.309
Short Cone	0.157	0.302	0.103	0.897	0.148	0.0	2.407
Long Cone	0.041	0.061	0.025	0.975	0.035	0.0	1.615
Full Cone	0.135	0.222	0.084	0.916	0.120	0.0	1.944

The dilators are calculated for a desired compatibility grade of 0.3.

Table IV. Model Fitting Results for Oblique Impact Data

Constant Model

Projectile	$\|C\|_X$	$\|K\|_F$	Discord	Comp. grade	Data Dilator	Model Dilators Minimal	Model Dilators Inclusive
Flat	2.138	4.437	1.443	0.0	2.062	4.265	8.417
Hemispherical	2.099	7.843	1.656	0.0	2.366	7.468	14.941
Short Cone	1.118	2.322	0.755	0.245	1.078	1.241	5.392
Long Cone	1.762	3.656	1.189	0.0	1.698	3.148	7.299
Full Cone	2.662	5.525	1.796	0.0	2.566	5.817	9.968

Linear Model

Projectile	$\|C\|_X$	$\|K\|_F$	Discord	Comp. grade	Data Dilator	Model Dilators Minimal	Model Dilators Inclusive
Flat	0.491	0.611	0.272	0.728	0.380	0.0	1.889
Hemispherical	0.913	1.950	0.622	0.378	0.905	0.690	5.464
Short Cone	0.541	1.066	0.359	0.641	0.518	0.0	3.697
Long Cone	1.071	2.041	0.702	0.298	1.027	1.081	5.283
Full Cone	0.445	0.495	0.235	0.765	0.328	0.0	1.641

The dilators are calculated for a desired compatibility grade of 0.3.

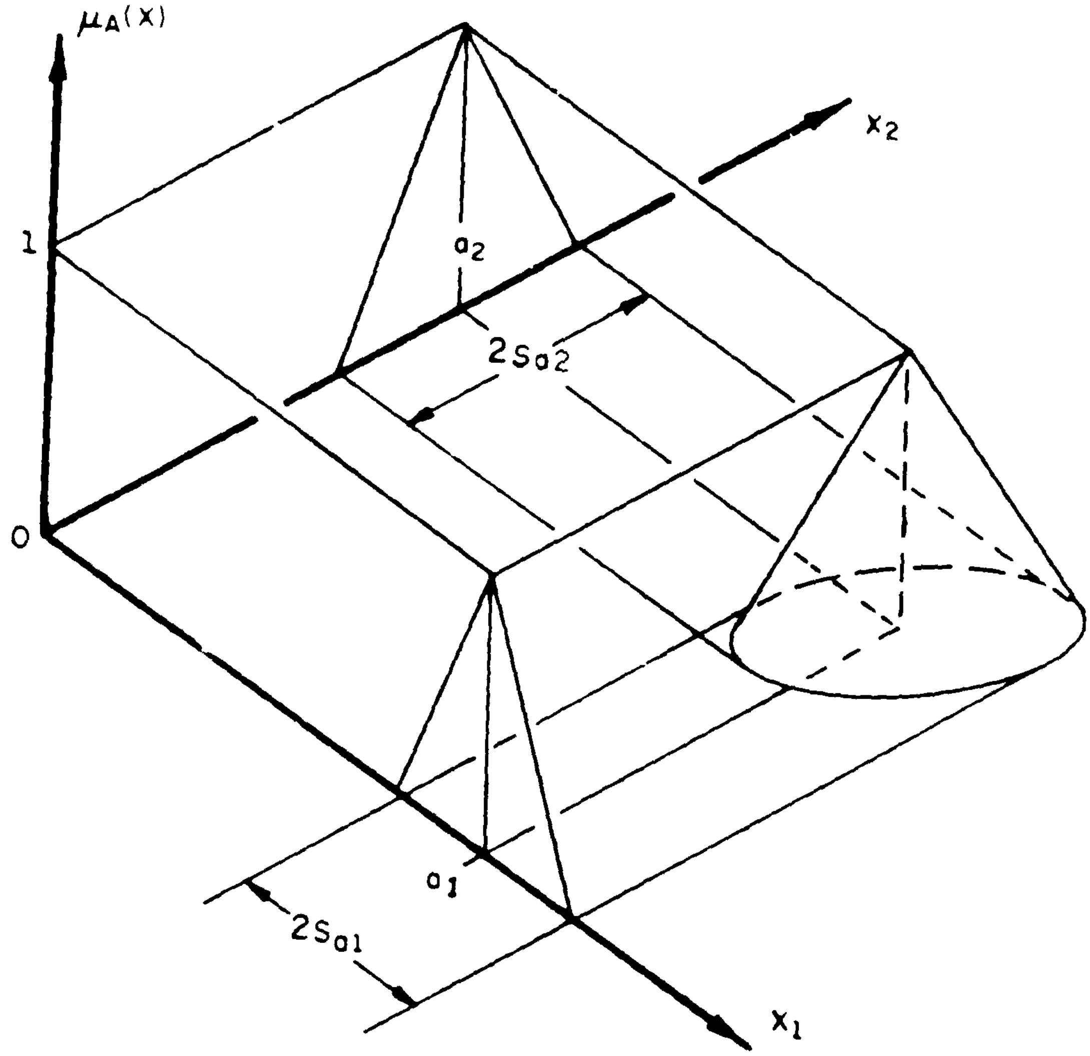

Figure 1. Conical Membership Function in Two Dimensions

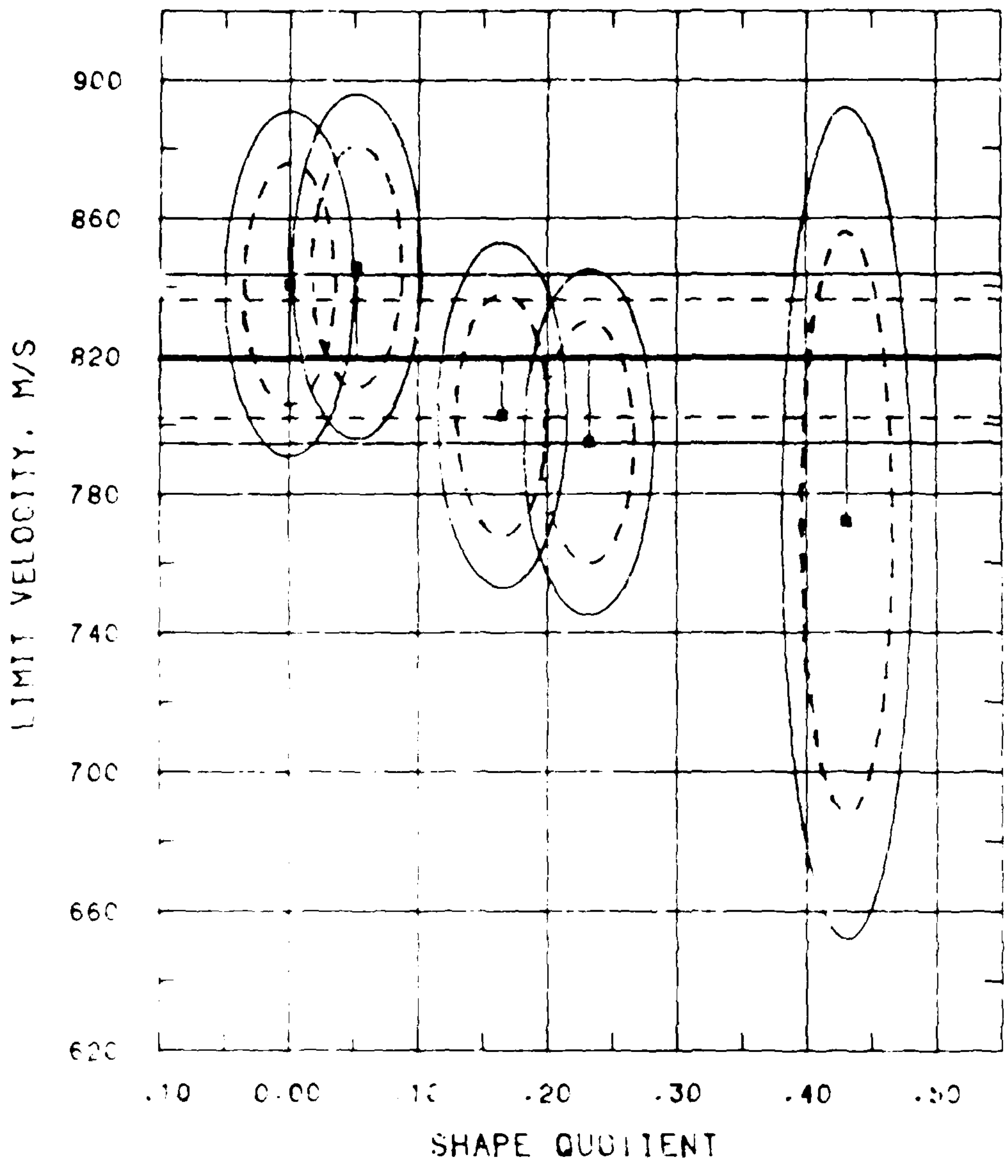

Figure 2. Constant Model Fitted to Normal Impact Data

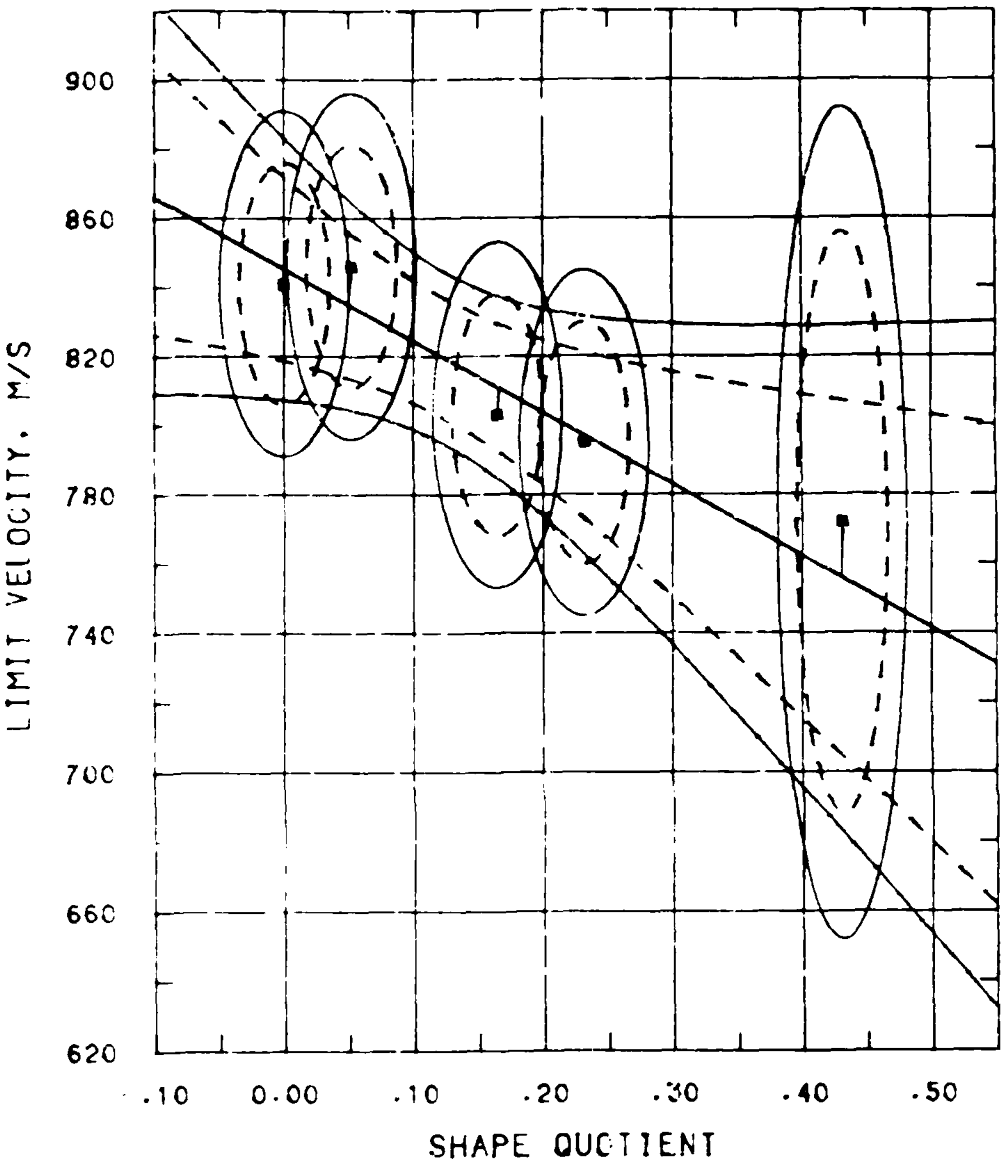

Figure 3. Linear Model Fitted to Normal Impact Data

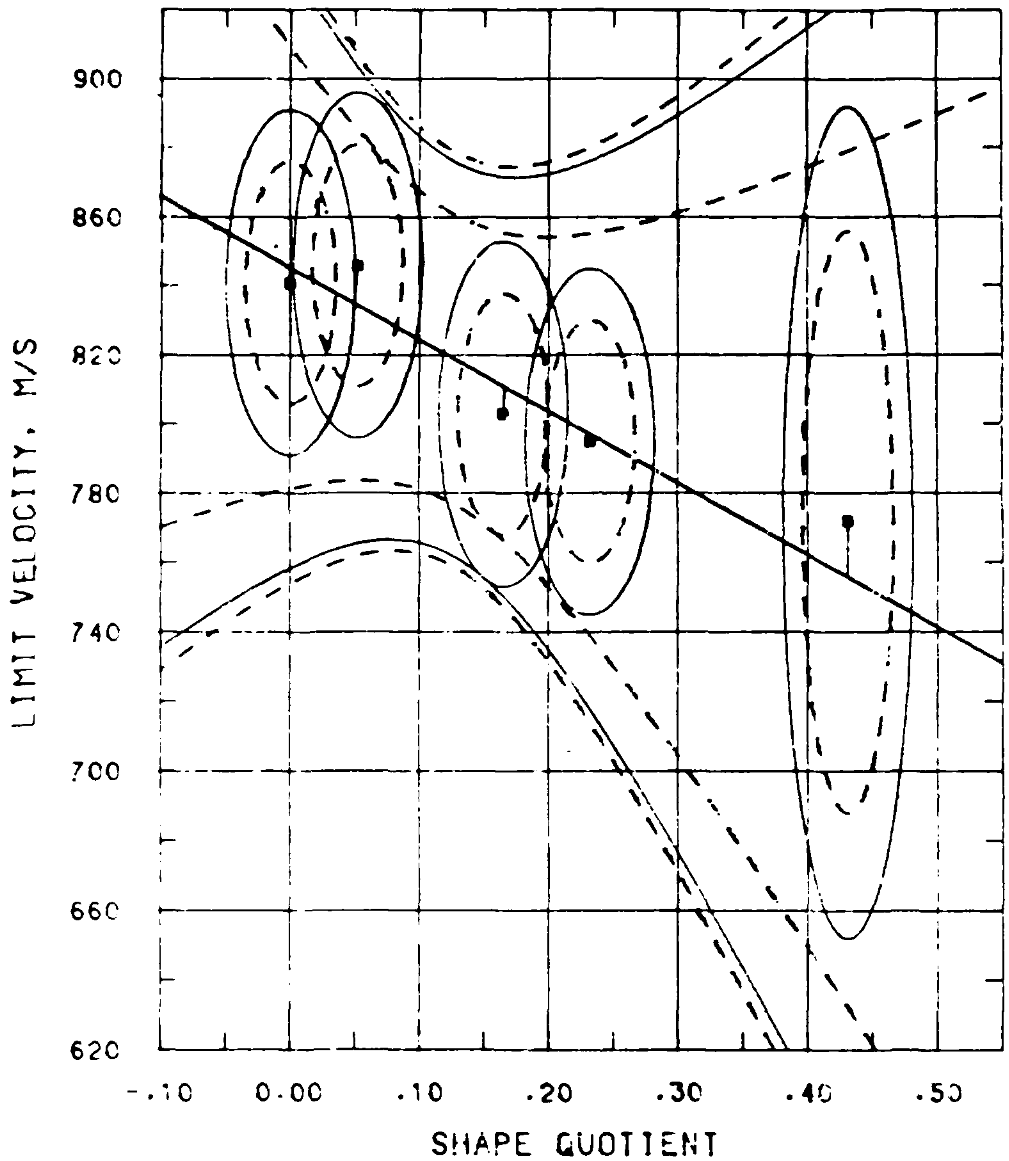

Figure 4. Linear Model with Dilated Spread Fitted to Normal Impact Data

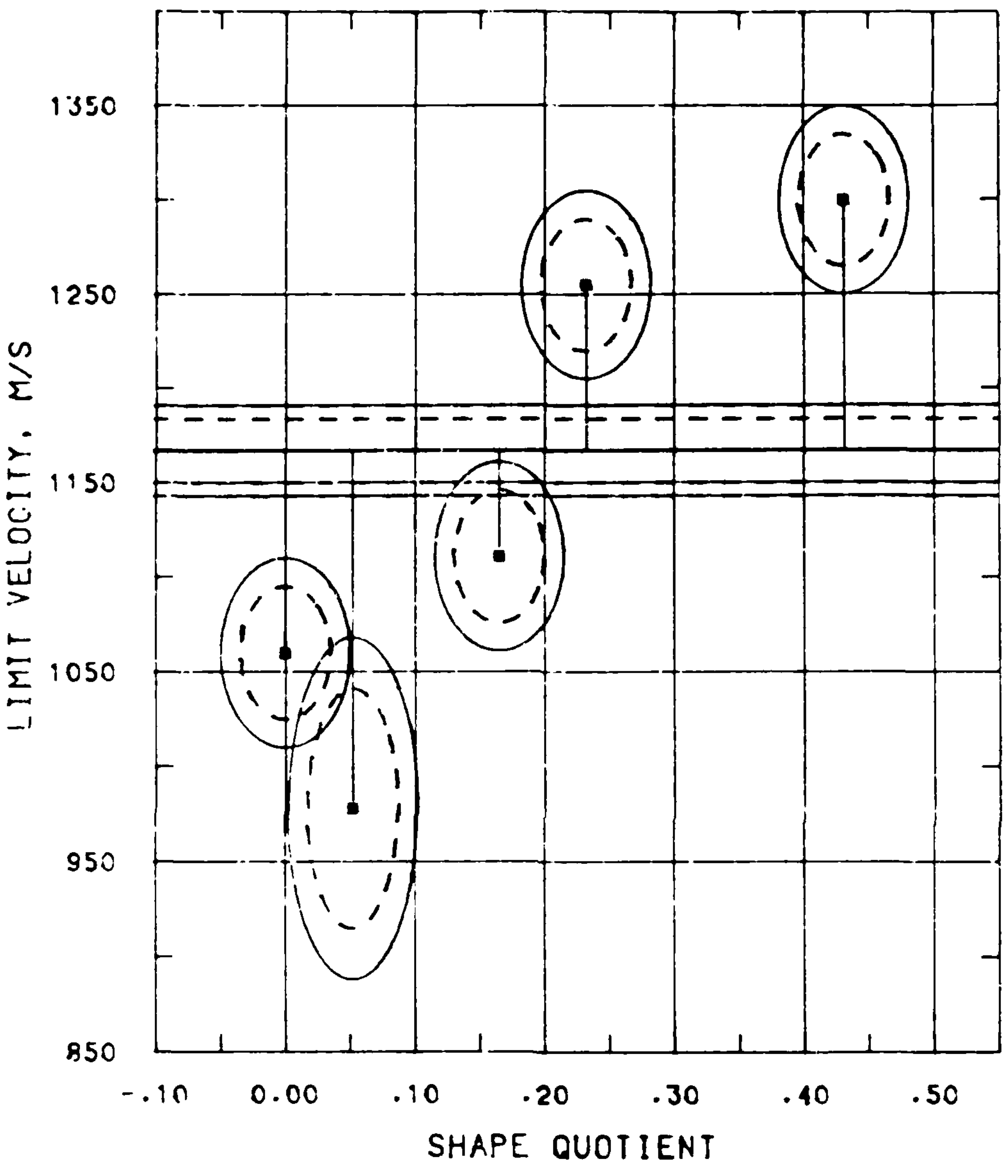

Figure 5. Constant Model Fitted to Oblique Impact Data

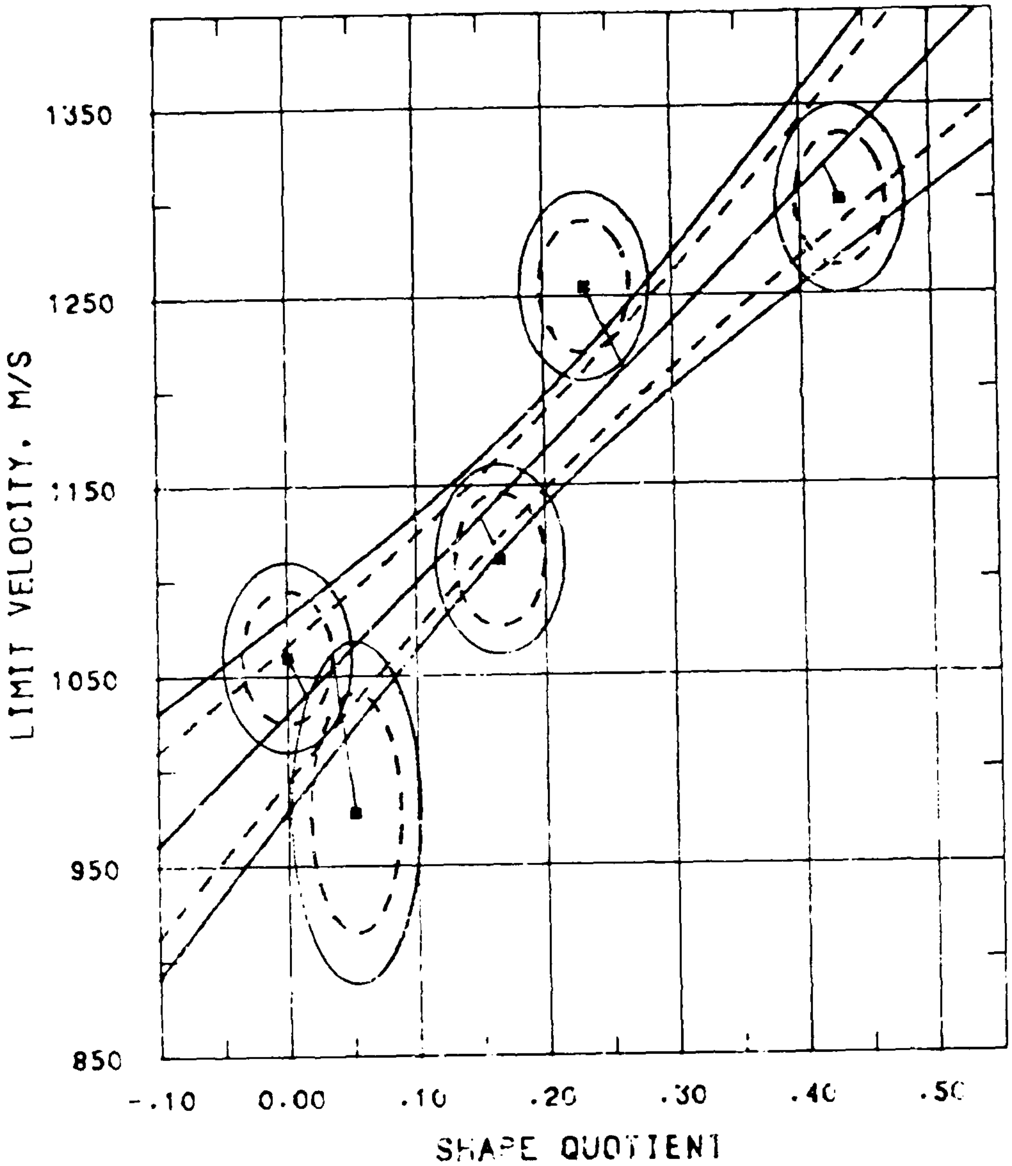

Figure 6. Linear Model Fitted to Oblique Impact Data

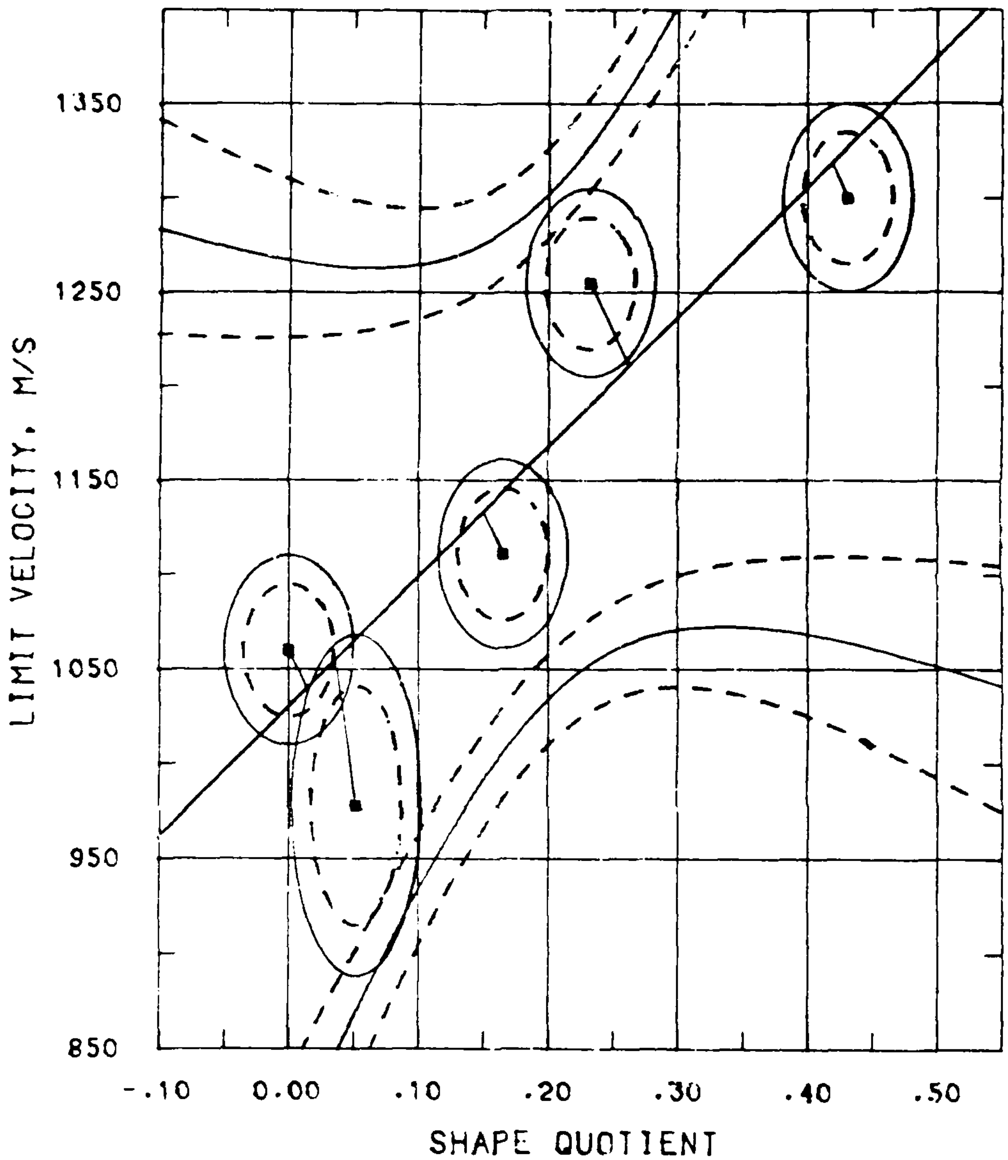

Figure 7. Linear Model with Dilated Spread Fitted to Oblique Impact Data

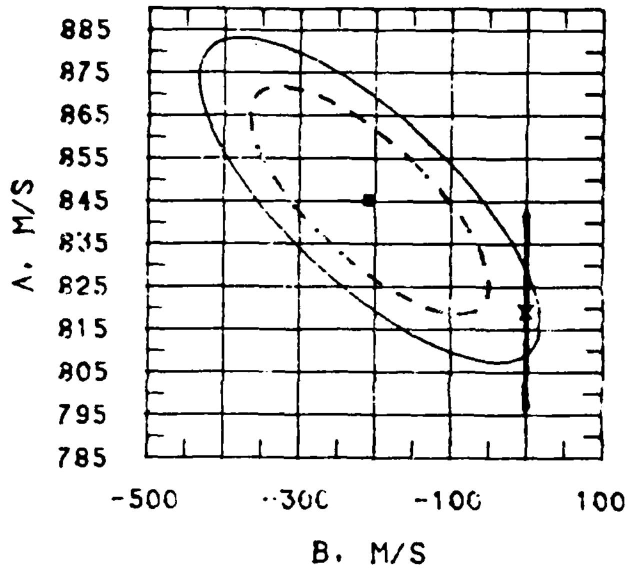

Figure 8. Model Parameters for Normal Impact Data

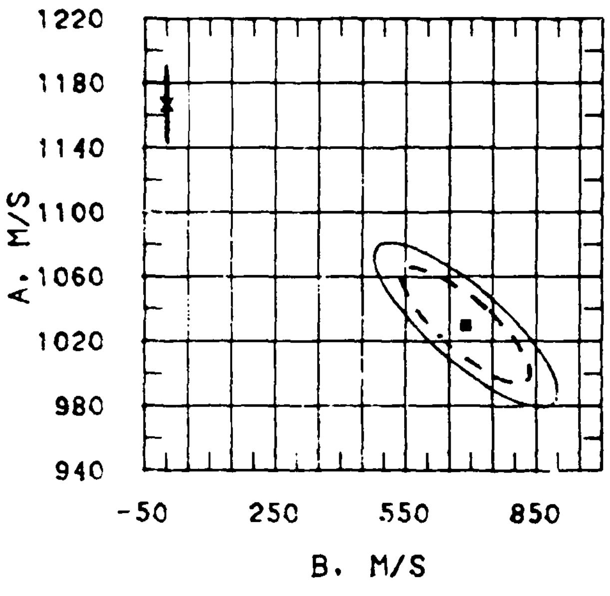

Figure 9. Model Parameters for Oblique Impact Data

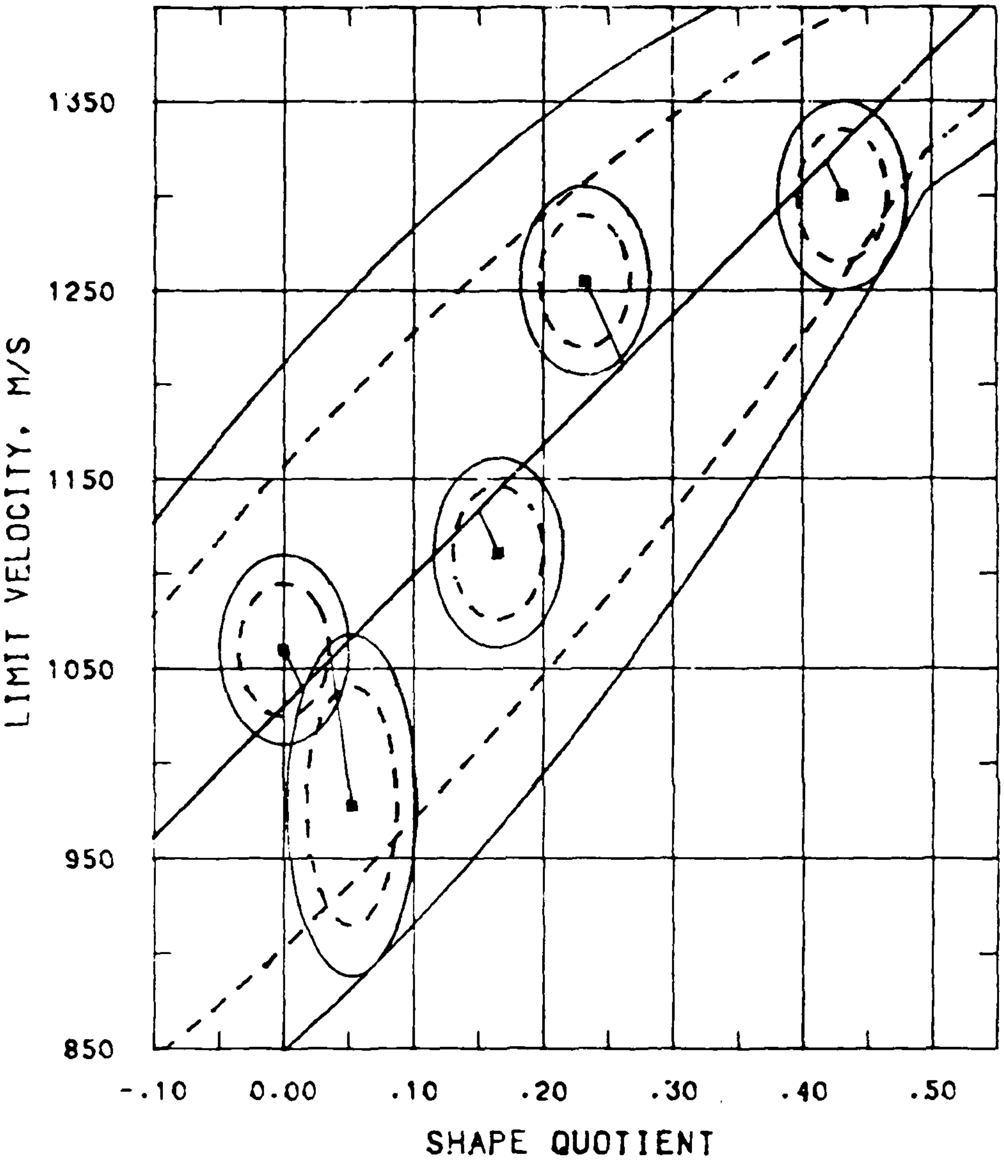

Figure 10. Linear Model with Modified Spread Fitted to Oblique Impact Data

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